APPLICATION OF THE REGULARIZATION PRINCIPLE TO THE FORMULATION OF APPROXIMATE SOLUTIONS OF INVERSE HEAT-CONDUCTION PROBLEMS

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The synthesis of a Tikhonov-regularizing algorithm is discussed for the solution of integral equations in inverse heat-conduction problems of the first kind.

Many inverse heat-conduction problems entailing the determination of transient boundary conditions for constant thermophysical characteristics of a solid require the solution of linear Volterra integral equations of the first kind:

$$A[u] \equiv \int_{\tau_0}^{\tau} u(\xi) K(\tau, \xi) d\xi = f_\delta(\tau), \quad \tau \leqslant \tau, \ \xi \leqslant \tau_m, \tag{1}$$

where $u(\tau)$ is the desired solution (unknown heat flux, temperature at the boundary of the body, or thermal potential density) and f_δ is the input function, which is known with a certain approximation δ.

As an example of this kind of problem we consider the determination of the heat flux admitted to a semi-infinite body having a moving boundary and a zero-valued initial temperature distribution:

$$\frac{\partial T(x, \tau)}{\partial \tau} = a \frac{\partial^2 T(x, \tau)}{\partial x^2}, \quad x > X(\tau), \quad \tau > \tau_0, \tag{2}$$

$$T(x, \tau_0) = T_0(x), \tag{3}$$

$$T(x_1, \tau) = f(\tau), \tag{4}$$

$$\left.\begin{aligned} \lambda \frac{\partial T(X(\tau), \tau)}{\partial x} + q(\tau) = 0, \\ \lambda \frac{\partial T(\infty, \tau)}{\partial x} = 0, \end{aligned}\right\} \tag{5}$$

where $X(\tau)$ is the known law governing the motion of the boundary (a continuous function), $T_0(x)$ is a specified continuous differentiable function, and $q(\tau)$ is an unknown function.

We reduce the problem (2)-(5) to a zero-valued boundary condition. To do so we consider a function $\varphi(x)$, $x \geq C_1$ [$C_1 < X(\tau)$], continuous together with its derivative, such that $\varphi(x) \equiv T_0(x)$ for $x \geq X(\tau_0)$ ($\partial T_0/\partial x \to 0$ as $x \to \infty$).

It is well known [1] that the expression

$$z(x, \tau) = \int_{C_1}^{\infty} \frac{1}{\sqrt{4\pi a\tau}} \exp\left[-\frac{(x-\eta)^2}{4a\tau}\right] \varphi(\eta) d\eta$$

is a solution of Eq. (2) satisfying the initial condition.

The reduced problem is now stated as follows:

$$w(x, \tau) = T(x, \tau) - z(x, \tau),$$

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$$\frac{\partial w(x,\tau)}{\partial\tau} = a\frac{\partial^2 w(x,\tau)}{\partial x^2}, \quad x > X(\tau), \quad \tau > \tau_0,$$

$$w(x, \tau_0) = 0,$$

$$w(x_1, \tau) = f(\tau) - z(x_1, \tau),$$

$$\lambda\frac{\partial w(X(\tau),\tau)}{\partial x} + q(\tau) + \lambda\frac{\partial z(X(\tau),\tau)}{\partial x} = 0,$$

$$\lambda\frac{\partial w(\infty,\tau)}{\partial x} = 0.$$

We represent the function $w(x, \tau)$ by means of the thermal potential of a simple layer:

$$w(x,\tau) = \frac{a}{2\sqrt{\pi}}\int_{\tau_0}^{\tau}\nu(\xi)\frac{1}{\sqrt{a(\tau-\xi)}}\exp\left[-\frac{(x-X(\xi))^2}{4a(\tau-\xi)}\right]d\xi,$$

where $\nu(\tau)$ is the thermal potential density.

The derivative $\partial w(x, \tau)/\partial x$ is discontinuous at the boundary $x = X(\tau)$:

$$\frac{\partial w(X(\tau)+0,\tau)}{\partial x} = \frac{\partial w(X(\tau),\tau)}{\partial x} - \frac{\nu(\tau)}{2}$$

and its limiting value is determined by the boundary condition. As a result

$$q(\tau) = -\frac{\partial z(X(\tau),\tau)}{\partial x} + \frac{\lambda}{2}\left\{\frac{\nu(\tau)}{2} + \int_{\tau_0}^{\tau}\nu(\xi)\frac{X(\tau)-X(\xi)}{2\sqrt{a\pi(\tau-\xi)^3}}\right.$$

$$\left.\times\exp\left[-\frac{(X(\tau)-X(\xi))^2}{4a(\tau-\xi)}\right]d\xi\right\}.$$

The potential density $\nu(\tau)$ must be determined from the known function $f(\tau)$ by solving the following Volterra integral equation of the first kind:

$$\frac{a}{2\sqrt{\pi}}\int_{\tau_0}^{\tau}\nu(\xi)\frac{1}{\sqrt{a(\tau-\xi)}}\exp\left[-\frac{(x_1-X(\xi))^2}{4a(\tau-\xi)}\right]d\xi$$

$$= f(\tau) - z(x_1, \tau).$$

This problem is the most complex link in the reconstruction of the heat flux in a semi-infinite body with a moving boundary. The required function $\nu(\tau)$ is unstable under perturbations of the right-hand side due to the incorrectness of the given problem. On the other hand, inasmuch as the right-hand side is known with a certain approximation, the solution $\nu(\tau)$ must be consistent with the accuracy with which the input data are specified.

An analogous problem is met in the solution of other linear inverse heat-conduction problems [2-6].

We now synthesize a regularizing algorithm for the solution of Eq. (1). We base it on the general deviation principle developed by Morozov for the solution of operator equations by the regularization method [7, 8].

Assuming that the required function has a definite smoothness, we adopt as the admissible class of solutions the Sobolev space $W_2^n[\tau_0, \tau_m]$ of generalized-differentiable functions. We assume that the input data belong to the space $L_2[\tau_0, \tau_m]$ of functions integrable in the square. We also assume that the integral operator with kernel $K(\tau, \xi)$ is defined on the entire space W_2^n and that an operator $A_{\Delta\tau}[u]$ uniformly approximating the initial operator has been constructed, i.e., that the following condition holds for the norm of the difference of the operators:

$$\|A_{\Delta\tau} - A\|_{L_2} \leqslant \varkappa(\Delta\tau),$$

where $\varkappa(\Delta\tau) \to 0$ as $\Delta\tau \to 0$.

Finally, it is required that the following inequality hold for the deviation of the approximate input data from the "exact" data:

$$\int_{\tau_0}^{\tau_m} [\bar{f}(\tau) - f_\delta(\tau)]^2 d\tau < \delta.$$

We consider the expression

$$\Delta = \int_{\tau_0}^{\tau_m} \{A_{\Delta\tau}[\bar{u}(\tau)] - f_\delta(\tau)\}^2 d\tau,$$

in which $\bar{u}$ is the solution of (1) corresponding to the "exact" right-hand $\bar{f}$.

By the triangular rule we have

$$\Delta \leqslant \int_{\tau_0}^{\tau_m} \{A_{\Delta\tau}[\bar{u}(\tau)] - A[u(\tau)]\}^2 d\tau + \int_{\tau_0}^{\tau_m} \{A[u(\tau)]$$

$$- f_\delta(\tau)\}^2 d\tau \leqslant \varkappa(\Delta\tau) \int_{\tau_0}^{\tau_m} \bar{u}^2(\tau)\, d\tau + \delta.$$

Normally for the solution of inverse heat-conduction problems it is always possible to approximate the operator A with an accuracy such that

$$\varkappa(\Delta\tau) \ll \delta.$$

We assume hereinafter, therefore, that

$$\Delta \leqslant \delta. \tag{6}$$

Regarding (6) as a condition generating a set of formal solutions $u \in W_2^n$ of (1), we segregate from it the function $u_{\delta\Delta\tau}$ realizing the lower bound of the functional

$$\inf \|u - u^*\|^2_{W_2^n} = \int_{\tau_0}^{\tau_m} \Big[\sum_{i=0}^{n} k_i(\tau)\,(u - u^*)^{(i)2}\Big] d\tau. \tag{7}$$

Equation (7) corresponds to the minimum deviation of the required element $u_{\delta\Delta\tau}$ from the element u^* specified in W_2^n metric.

The operator A[u] is linear and continuous, so that the stated problem is uniquely solvable [7, 9]. The sequence of approximate solutions in this case converges strongly to the exact solution of (1):

$$\lim_{\delta,\Delta\tau \to 0} u_{\delta\Delta\tau} = \bar{u}$$

in $W_2^n[\tau_0, \tau_m]$ whence we also infer their uniform convergence together with the derivatives of order up to and including (n—1).

The formulated quadratic programming problem (6)-(7) is reduced to a problem in the form of the regularization method of A. N. Tikhonov if we allow [9] for the fact that the function $u_{\delta\Delta\tau}$ corresponds to the strict equality in condition (6).

Invoking the method of Lagrange multipliers, we arrive at the following variational problem:

$$\min_{u \in W_2^n} \Big\{\Phi_\alpha[u, f_\delta] = \int_{\tau_0}^{\tau_m} \Big[\int_{\tau_0}^{\tau} K(\tau, \xi)\, u(\xi)\, d\xi - f_\delta(\tau)\Big]^2 d\tau$$

$$+ \alpha \int_{\tau_0}^{\tau_m} \Big[\sum_{i=0}^{n} k_i(\xi)\,[u(\xi) - u^*(\xi)]^{(i)2}\Big] d\xi\Big\}, \tag{8}$$

in which $\alpha = 1/\lambda$ (λ is a Lagrange multiplier) is determined from the condition

$$\int_{\tau_0}^{\tau_m} \Big\{\int_{\tau_0}^{\tau} K(\tau, \xi)\, u_\alpha(\xi)\, d\xi - f_\delta(\tau)\Big\}^2 d\tau = \delta^2. \tag{9}$$

Expanding the difference squared in the first term of (8) and changing the order of integration over the triangular domain, we obtain

$$\Phi_\alpha[u, f_\delta] = \int_{\tau_0}^{\tau_m}\Big\{\int_{\xi}^{\tau_m} d\tau \int_{\tau_0}^{\tau} K(\tau, \xi) K(\tau, \zeta) u(\xi) u(\zeta) d\zeta$$

$$- 2\int_{\xi}^{\tau_m} f_\delta(\tau) K(\tau, \xi) u(\xi) d\tau + \alpha \sum_{i=0}^{n} k_i(\xi) [u(\xi) - u^*(\xi)]^{(i)2}\Big\} d\xi$$

$$+ \int_{\tau_0}^{\tau_m} f_\delta^2(\tau) d\tau. \tag{10}$$

We write the Euler equation for (10). We have as a result

$$\int_{\xi}^{\tau_m} d\tau \int_{\tau_0}^{\tau} K(\tau, \xi) K(\tau, \zeta) u(\zeta) d\zeta + \int_{\xi}^{\tau_m} d\tau \int_{\tau_0}^{\tau} K(\tau, \xi) K(\tau, \zeta) u(\xi) d\zeta$$

$$- 2\int_{\xi}^{\tau_m} f_\delta(\tau) K(\tau, \xi) d\tau + 2\alpha \Big\{\sum_{i=0}^{n} (-1)^i [k_i(\xi) (u(\xi) - u^*(\xi))^{(i)}]^{(i)}\Big\} = 0.$$

Noting that the domain of integration for the multiple integrals has the form of a rectangular trapezoid and changing the order of integration, we finally obtain

$$\int_{\tau_0}^{\tau} \overline{K}_1(\xi, \zeta) u(\zeta) d\zeta + \int_{\xi}^{\tau_m} \overline{K}_2(\xi, \zeta) u(\zeta) d\zeta - \overline{b}(\xi)$$

$$+ \alpha \sum_{i=0}^{n} (-1)^i [k_i(\xi) (u(\xi) - u^*(\xi)^{(i)}]^{(i)} = 0, \tag{11}$$

where

$$\overline{K}_1(\xi, \zeta) = \int_{\xi}^{\tau_m} K(\tau, \xi) K(\tau, \zeta) d\tau,$$

$$\overline{K}_2(\xi, \zeta) = \int_{\zeta}^{\tau_m} K(\tau, \xi) K(\tau, \zeta) d\tau,$$

$$\overline{b}(\xi) = \int_{\xi}^{\tau_m} f_\delta(\tau) K(\tau, \xi) d\tau.$$

The boundary conditions for the solution of (11) are

$$u(\tau_0) = u_0, \quad u'(\tau_0) = u_0', \ \ldots, \ u^{(n-1)}(\tau_0) = u_0^{(n-1)};$$

$$u(\tau_m) = u_m, \quad u'(\tau_m) = u_m', \ \ldots, \ u^{(n-1)}(\tau_m) = u_m^{(n-1)}.$$

If it is too difficult to specify the zeroth approximation to the required heat flux $q(\tau)$ [the surface temperature $T_W(\tau)$ or the thermal potential density $\nu(\tau)$], we can set $u^*(\tau) = 0$. In this case we obtain the primary regularized solution $u_{1\alpha}(\tau)$. To obtain the next-higher approximation we logically use $u_{1\alpha}(\tau)$ as $u^*(\tau)$. We then obtain the secondary regularized solution $u_{2\alpha}(\tau)$, and so on.

The upper summation limit in the second integral terms of (11) corresponds to the regularization order. If $n = 0$ ($W_2^0 = L_2$), then the square deviation of the required solution from the given approximation is minimized. In this case the convergence of the regularized approximations does not necessarily imply their uniform convergence. In general, the number n and the functions $k_i(\xi)$ have to be selected on the basis of the specifics of the particular inverse heat-conduction problem as well as the existing a priori information about the nature of the required function. It could be required, for example, that the solution have reasonable smoothness in the sense of minimizing the expression [4, 10].

$$\int_{\tau_0}^{\tau_m} u'(\tau)^2 d\tau \quad \text{or} \quad \int_{\tau_0}^{\tau_m} u''(\tau)^2 d\tau.$$

The choice of approximation to the required solution by the deviation principle requires knowledge of the error of the initial data. In L_2 metric

$$\delta^2 = \int_{\tau_0}^{\tau_m} \sigma^2(\tau)\, d\tau,$$

where $\sigma(\tau)$ is the rms error of the right-hand side of (1).

If σ varies appreciably with time, it may prove convenient (for example, so as not to "oversmooth" the solution) to find the regularized approximations by intervals, in which certain average values of σ_i are delimited:

$$u(\tau) = \begin{cases} u_{\alpha 1}(\tau) & \tau_0 \leqslant \tau \leqslant \tau_1 \quad \alpha_1 = f(\sigma_1), \\ u_{\alpha 2}(\tau) & \tau_1 \leqslant \tau \leqslant \tau_2 \quad \alpha_2 = f(\sigma_2), \\ \vdots & \vdots \qquad\qquad \vdots \\ u_{\alpha k}(\tau) & \tau_{k-1} \leqslant \tau \leqslant \tau_k \quad \alpha_k = f(\sigma_k). \end{cases}$$

We now consider certain aspects of the computer implementation of the given method for the solution of (1) in application to inverse heat-conduction problems.

Writing the finite-difference equivalent for Eq. (11) in the general case and solving it for known boundary conditions, we determine the grid function $u_\alpha^{\Delta\tau}(\tau)$. However, a more efficient computation algorithm can be obtained if we use for regularization the approximation solution constructed for the direct problem of computing $T(x_1, \tau)$ from a known function $u(\tau)$ with allowance for the possibility of integrating $K(\tau, \xi)$ analytically on a selected interval $(\tau_{i-1}, \tau_i]$.

Thus, for problem (10) putting $\tau_0 = 0$, we can approximately write [7]

$$\sum_{i=1}^{n} \varphi_i^n \bar{\nu}_i = F_n, \quad n = 1,\ 2,\ \ldots,\ m, \tag{12}$$

where

$$\bar{\nu}_i = \frac{\nu_i + \nu_{i-1}}{2}, \quad F_n = f(\tau_n) - z(x_1, \tau_n),$$

$$\varphi_i^n = \sqrt{a\Delta\tau} \left\{ \sqrt{n-p}\, i\Phi^* \left[\frac{x - \bar{X}_i}{2\sqrt{a\Delta\tau (n-p)}} \right] \right\}_{p=i}^{p=i-1}$$

Considering the first-order regularization n = 1 for (12) and choosing $k_0 = 0$ and $k_1 = 1$, we write the finite-difference form of the functional (8):

$$\Phi_\alpha^{\Delta\tau} [\bar{\nu}, F] = \sum_{n=1}^{m} \left\{ \sum_{i=1}^{n} \varphi_i^n \bar{\nu}_i - F_n \right\}^2 \Delta\tau + \alpha \sum_{i=1}^{n} \frac{(\bar{\nu}_{i-1} - \bar{\nu}_i)^2}{\Delta\tau}. \tag{13}$$

Setting the derivative with respect to $\bar{\nu}_i$ equal to zero and assigning the boundary conditions

$$\bar{\nu}'(0) \sim \frac{\bar{\nu}_1 - \bar{\nu}_0}{\Delta\tau} = 0, \quad \bar{\nu}'(\tau_m) \sim \frac{\bar{\nu}_{m+1} - \bar{\nu}_m}{\Delta\tau} = 0,$$

we arrive at the following system of linear algebraic equations with a symmetrical positive definite matrix:

$$\sum_{l=1}^{m} a_{lk} \bar{\nu}_l = F_k, \quad k = 1,\ 2,\ \ldots,\ m, \tag{14}$$

in which

$$a_{lk} = \Delta\tau^2 \sum_{n=l}^{m} \varphi_k^n \varphi_l^n, \quad l \geqslant k+2,$$

$$a_{lk} = \Delta\tau^2 \sum_{n=l}^{m} \varphi_k^n \varphi_l^n - \alpha, \quad l = k+1,$$

$$a_{lk} = \Delta\tau^2 \sum_{n=l}^{m} (\varphi_l^n)^2 + 2\alpha, \quad l \neq 1;\ m,$$

$$a_{ll} = \Delta\tau^2 \sum_{n=l}^{m} (\varphi_l^n)^2 + \alpha, \quad l = 1;\ m,$$

$$F_k = \sum_{n=k}^{n} b_{kn} F_n, \quad b_{kn} = \Delta\tau^2 \varphi_k^n.$$

The number α corresponding to the best approximation of the solution in the sense of a deviation in the form of Eq. (9) is found from the condition

$$\min_{\alpha} \left\{ \Delta \sim \left| \left[\sum_{n=1}^{m} \left(\sum_{i=1}^{n} \varphi_i^n \bar{v}_{\alpha i} - F_n \right)^2 \Delta\tau \right]^{1/2} - \delta \right| \right\}.$$

The arbitrariness in the designation of the boundary conditions for the minimization of (13) necessarily distorts the form of the required function at the end points of the given time interval. However, as numerical experiments have shown, that distortion is confined to relatively small neighborhoods of τ_0 and τ_m.

NOTATION

A is an integral operator;
a is the thermal diffusivity;
f is the set of input data;
q is the heat flux;
T is the absolute temperature;
u is the solution of the integral equation;
u* is a trail solution;
X is the coordinate of the moving boundary of the body;
x is the instantaneous coordinate;
α is the regularization parameter;
δ is the error of the input data;
$\Delta\tau$ is the time step;
λ is the thermal conductivity;
ν is the thermal potential density;
τ is the time.

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